# Cardinal invariants associated with some combinatorial statements (Studies in Relative Consistency Proofs with Particular Emphasis on Set Theoretic Methods)

数理解析研究所講究録 1304 巻 2003 年 78-87



# Cardinal invariants associated with some combinatorial statements

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**Abstract**

T. Bartoszyński [1] characterized the uniformity $\mathbf{non}(\mathcal{M})$ of the meager ideal on the real line as the smallest size of a family $X \subset \omega^\omega$ such that $\forall y \in \omega^\omega\ \exists x \in X\ \exists^\infty n < \omega\ y(n) = x(n)$. By replacing $\omega^\omega$ by certain restricted subsets, we can get weaker combinatorial statements and define cardinal invariants. In this talk, we study these cardinal invariants.

## 0 Introduction

We use standard notion and notations in set theory (see e.g. [3]). Set

$$\mathcal{F} = \{ f \in (\omega \setminus \{0\})^\omega \mid f \text{ is non-decreasing and } \lim_{n<\omega} f(n) = \omega \}.$$

For each $f \in \mathcal{F}$, define the cardinal $\theta_f$ by

$$\theta_f = \min\{ |X| \mid X \subset \prod_{n<\omega} f(n) \text{ and } \forall y \in \prod_{n<\omega} f(n)\ \exists^\infty n < \omega\ y(n) = x(n) \}.$$

By the Bartoszyński's characterization of $\mathbf{non}(\mathcal{M})$, it holds that $\theta_f \leq \mathbf{non}(\mathcal{M})$ for all $f \in \mathcal{F}$. Also, it is easy to see that $\theta_{f_1} \leq \theta_{f_2}$ if $f_1,\ f_2 \in \mathcal{F}$ and $f_1 \leq^* f_2$. In the next section, we show that, in a certain generic model which is obtained by adjoining random reals, $\theta_{f_1} < \theta_{f_2}$ holds for some $f_1,\ f_2 \in \mathcal{F}$. Put $\theta = \min\{ \theta_f \mid f \in \mathcal{F} \}$. Let me introduce another cardinal invariant $\theta^*$ which is associated with a weaker combinatorial statement. For this, we need some definitions. Set

$$\mathcal{H} = \{ h \in \omega^\omega \mid h \text{ is strictly increasing and } \lim_{n<\omega} h(n+1) - h(n) = \omega \}.$$

For each $h \in \mathcal{H}$ and $n < \omega$, $a_n^h$ denotes the interval $[h(n), h(n+1))$ of $\omega$. Define $\theta^*$ by

$$\theta^* = \min\{ |W| \mid W \subset 2^\omega \times \mathcal{H} \text{ and } \forall y \in 2^\omega\ \exists (x,h) \in W\ \exists^\infty n < \omega\ y \restriction a_n^h = x \restriction a_n^h \}.$$

It is easy to check that $\omega_1 \leq \theta^* \leq \theta$. Furtheremore, we have:

**Theorem 0.1** *Assume that* $\mathrm{cof}([\mathbf{d}]^\omega, \subset) = \mathbf{d}$. *Then, it holds that* $\theta^* \leq \mathbf{d}$.

**Proof** Take a sufficiently large regular cardinal $\rho$. By using the assumption, take an elementary substructure $M$ of $H(\rho)$ such that

$M \cap \omega^\omega$ is a dominating family and $|M| = \mathbf{d}$ and $M \cap [M]^\omega$ is $\subset$-cofinal in $[M]^\omega$.

Since $M \cap \omega^\omega$ is a dominating family, it holds that

(*) $\quad \forall h \in \mathcal{H}\, \exists h' \in M \cap \mathcal{H}\, \forall^\infty n < \omega\, \exists m < \omega \;\; a^{h'}_m \subset a^h_n$.

We show that $W = M \cap (2^\omega \times \mathcal{H})$ satisfy the definition of $\theta^*$. To get a contradiction, assume that there is $y \in 2^\omega$ such that

$\forall^\infty n < \omega \;\; y \restriction a^h_n \neq x \restriction a^h_n$, for all $(x, h) \in W$.

Put $X = 2^\omega \cap M$. The next claim is easily verfied by using (*).

**Claim 0.2** $\quad \forall x \in X\, \exists k < \omega\, \forall^\infty m < \omega \;\; y \restriction [m, m+k) \neq x \restriction [m, m+k)$. $\quad \triangle$

By Claim 0.2, define $\varphi : X \to \omega$ by

$\varphi(x) =$ the largest $k < \omega$ such that $\exists^\infty m < \omega \;\; x \restriction [m, m+k) \subset y$.

It is easy to check that $\sup\{\, \varphi(x) \mid x \in X \,\} = \omega$. By this, since $[M]^\omega \cap M$ is $\subset$-cofinal in $[M]^\omega$, we can take $A = \{\, a_i \mid i < \omega \,\} \in M$ such that $\sup\{\, \varphi(a_i) \mid i < \omega \,\} = \omega$. Take $\psi : \omega \times \omega \to \omega$ such that, for each $(i, n) \in \omega \times \omega$,

$i + n + \varphi(a_i) \leq \psi(i, n)$ and $\exists m \in [n, \psi(i,n) - \varphi(a_i)) \;\; a_i \restriction [m, m + \varphi(a_i)) \subset y$.

Without loss of generality, we may assume that $\psi \in M$. Define $\langle\, k_i \mid i < \omega \,\rangle \in M$ by

$$\begin{cases} k_0 &= 0 \\ k_{i+1} &= \psi(i, k_i), \text{ for } i < \omega \end{cases}$$

and set $x = \bigcup_{i<\omega} a_i \restriction [k_i, k_{i+1}) \in X$. Then, it holds that

$\forall k < \omega\, \exists m < \omega \; x \restriction [m, m+k) \subset y$.

But this contradicts Claim 0.2 $\quad \square$

Let $\mathbf{C}_\omega$ be the notion of forcing which adds a Cohen real. Then, it holds that

$\Vdash_{\mathbf{C}_\omega} \forall y \in 2^\omega\, \exists x \in 2^\omega \cap \mathbf{V}\, \exists^\infty n < \omega \;\; x \restriction [n^2, n^2+n) = y \restriction [n^2, n^2+n)$.

So, $\theta^* < \mathbf{d}$ holds in a certain Cohen generic model.

It is known that the assumption $\mathrm{cof}([\mathbf{d}]^\omega, \subset) = \mathbf{d}$ is followed from the non-existence of $0^\#$. So, it seems to prove Theorem 0.1 without this assumption. But I failed to find a proof.

**Question 0.1** *Is $\theta^* \leq \mathbf{d}$ proved in* ZFC*?*

In sections 2, 3, 4, we show that the cardinals $\omega_1$, $\theta$, $\theta^*$ can be separated for certain generic models.

# 1 Generic extensions by random reals

For each infinite cardinal $\kappa$, we denote by $\mathbf{B}(\kappa)$ the measure algebra which adds a random function from $\kappa$ to 2 and by $\mu_\kappa : \mathbf{B}(\kappa) \to [0,1]$ the measure function. In this section, we prove the following theorem.

**Theorem 1.1** *Assume* CH. *Let $\kappa > \omega_1$ be a regular cardinal. Then, there are $f_1, f_2 \in \mathcal{F}$ such that*

$\Vdash_{\mathbf{B}(\kappa)} \theta_{f_1} = \omega_1$ *and* $\theta_{f_2} = \kappa$.

Set $f_2 = \langle 2^n \mid n < \omega \rangle \in \mathcal{F}$. The next well-known lemma guarantees that this $f_2$ is as required in Theorem 1.1.

**Lemma 1.2** (Forklore) $\Vdash_{\mathbf{B}(\omega)} \exists y \in \prod_{n<\omega} f_2(n) \, \forall x \in \prod_{n<\omega} f_2(n) \cap \mathbf{V} \; \forall^\infty n < \omega \; x(n) \neq y(n)$.

**Proof** Define $k_n < \omega$ (for $n < \omega$) by

$k_0 = 0$ and $k_{n+1} = k_n + n$ for $n < \omega$.

For each $n < \omega$, put $I_n = [k_n, k_{n+1})$ and take a bijections from ${}^{I_n}2$ to $f_2(n)$. Using these bijections, we identify $\prod_{n<\omega} f_2(n)$ with $\prod_{n<\omega} {}^{I_n}2$. Let $\dot{g}$ be the canonical $\mathbf{B}(\omega)$-name of generic real. Define $\dot{y}$ by

$\Vdash \dot{y} = \langle \dot{g} \restriction I_n \mid n < \omega \rangle$.

It holds that, for each $n < \omega$ and $s : I_n \to 2$,

$\mu_\omega(\llbracket s = \dot{g} \restriction I_n \rrbracket) = 2^{-|I_n|} = 2^{-n}$.

So, $\mu_\omega(\llbracket \exists^\infty n < \omega \; x \restriction I_n = \dot{y}(n) \rrbracket) = 0$ for all $x \in 2^\omega$. This implies that

$\Vdash \forall^\infty n < \omega \; x(n) \neq \dot{y}(n)$, for all $x \in \prod_{n<\omega} {}^{I_n}2$. □

**Lemma 1.3** *Let $0 < K$, $M < \omega$. Suppose that $\{ b_i^m \mid i < K$ and $m < M \} \subset \mathbf{B}(\omega)$ and $b \in \mathbf{B}(\omega)$ satisfy*

$b = \sum_{i<K} b_i^m$, *for all* $m < M$.

*Then, there is a function $\varphi : M \to K$ such that*

$$\mu_\omega(\sum_{m<M} b_{\varphi(m)}^m) \geq \mu_\omega(b) - (\frac{K-1}{K})^M \mu_\omega(b).$$

**Proof** By induction on $M \in [1,\omega)$. The case $M = 1$ is clear. Let $M = M_0 + 1 > 1$. Using the induction hypothesis, take $\varphi_0 : M_0 \to K$ such that

$$\mu_\omega(\sum_{m<M_0} b^m_{\varphi_0(m)}) \geq \mu_\omega(b) - (\frac{K-1}{K})^{M_0}\mu_\omega(b).$$

Put $c = \sum_{m<M_0} b^m_{\varphi_0(m)}$. Since $b - c = \sum_{i<K}(b_i^{M_0} - c)$, there exists $j < K$ such that $\mu_\omega(b_j^{M_0}) \geq \frac{1}{K}\mu_\omega(b-c)$. Then, $\varphi = \varphi_0 ^\frown \langle j \rangle$ is as required. □

For each $n < \omega$, let

$$M_n = \min\{ M < \omega \mid (\frac{n}{n+1})^M < 2^{-n} \}.$$

Define $f_1 \in \mathcal{F}$ by

$$|\{ k < \omega \mid f_1(k) = n+1 \}| = M_n, \text{ for all } n < \omega.$$

The next lemma implies that $f_1$ satisfies the condition in Theorem 1.1.

**Lemma 1.4** $\Vdash_{\mathbf{B}(\omega)} \forall y \in \prod_{k<\omega} f_1(k)\, \exists x \in \prod_{k<\omega} f_1(k) \cap \mathbf{V}\ \exists^\infty k < \omega\ x(k) = y(k).$

**Proof** For each $n < \omega$, put $J_n = \{ k < \omega \mid f_1(k) = n+1 \}$. To show this lemma, let $\dot{y}$ be a $\mathbf{B}(\omega)$-name such that $\Vdash\ \dot{y} \in \prod_{k<\omega} f_1(k)$. For each $n < \omega$, using Lemma 1.3, take $s_n \in \prod_{k\in J_n} f_1(k)$ such that

$$\mu_\omega(\sum_{k\in J_n} [\![ s_n(k) = \dot{y}(k) ]\!]) \geq 1 - (\frac{n}{n+1})^{M_n}.$$

Put $x = \bigcup_{n<\omega} s_n$. It is easy to check that

$$\mu_\omega([\![ \forall^\infty n < \omega\, \exists k \in J_n\ x(k) = \dot{y}(k) ]\!]) = 0.$$

So, it holds that $\Vdash\ \exists^\infty k < \omega\ x(k) = \dot{y}(k)$. □

## 2 A forcing notion with the ccc which lifts up $\theta^*$

Define the forcing notion $(Q, \leq)$ by

$$Q \subset 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}$$

and, for any $(s, u) \in 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}$,

$$(s, u) \in Q$$

if and only if, for all $(x, h) \in u$ and all $k < \omega$,

if $a_k^h \setminus \mathrm{dom}(s) \neq \phi$ then $|a_k^h \setminus \mathrm{dom}(s)| \geq |u|$ or $\exists i \in a_k^h \cap \mathrm{dom}(s)\ x(i) \neq s(i)$,

and, for any $(s, u)$, $(s', u') \in Q$,

$$(s', u') \leq (s, u)$$

if and only if

$s' \supset s$ and $u' \supset u$ and, for all $(x, h) \in u$ and all $k < \omega$, if $a_k^h \cap (\mathrm{dom}(s') \setminus \mathrm{dom}(s)) \neq \phi$ then $|a_k^h \setminus \mathrm{dom}(s')| \geq |u'|$ or $\exists i \in a_k^h \cap \mathrm{dom}(s')\ x(i) \neq s'(i)$.

We show that a finite support iteration by the above forcing notion lifts up the value $\theta^*$. For this, we need several lemmas.

**Lemma 2.1** *Let $n < \omega$. Then, for every $(s,u) \in Q$, there is $s' \in 2^{<\omega}$ such that $(s',u) \in Q$ and $(s',u) \leq (s,u)$ and $n \in \mathrm{dom}(s)$.*

**Proof** For each $j < \omega$, define $\varphi_j : \mathcal{H} \to \omega$ by

$\varphi_j(h) =$ the unique $k < \omega$ such that $j \in a^h_k$.

For each $t \in 2^{<\omega}$, define $\psi_t : 2^\omega \times \mathcal{H} \to \omega$ by

$$\psi_t(x,h) = \begin{cases} |a^h_{\varphi_{\mathrm{dom}(t)}(h)} \setminus \mathrm{dom}(t)|, & \text{if } t \restriction a^h_{\varphi_{\mathrm{dom}(t)}(h)} \subset x, \\ |a^h_{\varphi_{\mathrm{dom}(t)}(h)+1}|, & \text{otherwise.} \end{cases}$$

To show this lemma, let $n < \omega$ and $(s,u) \in Q$. Put $m = \mathrm{dom}(s)$. Take $M < \omega$ such that

$n,\ m < M$ and $|a^h_{\varphi_M(h)} \setminus M| \geq |u|$, for all $(x,h) \in u$

By induction on $j \in [m, M]$, take $s_j : j \to 2$ as follows:

Put $s_m = s$. Suppose that $j \in [m, M)$ and $s_j$ has been defined. Let $l_j$ be the smallest element of $\{ \psi_{s_j}(x,h) \mid (x,h) \in u \}$. Take $(x_j, h_j) \in u$ such that $\psi_{s_j}(x_j,h_j) = l_j$. Set $s_{j+1} = s_j{}^\frown\langle 1 - x_j(j) \rangle$.

**Claim 2.2** $|\{ (x,h) \in u \mid \psi_{s_j}(x,h) < l \}| < l$, *for all $0 < l < \omega$ and $j \in [m, M]$*

$\because$) By induction on $j \in [m, M]$. The case $j = m$ is followed from the fact $(s,u) \in Q$. The case $j = j_0 + 1 > m$ is followed from the fact $\psi_{s_j}(x_{j_0}, h_{j_0}) \geq |u|$. $\triangle$

By Claim 2.2, it holds that $l_j > 0$, for all $j \in [m, M)$. So, it holds that $(s_M, u) \in Q$ and $(s_M, u) \leq (s,u)$. $\square$

**Lemma 2.3** *For each $(x,h) \in 2^\omega \times \mathcal{H}$,*

*$\{ (s,u) \in Q \mid (x,h) \in u \}$ is dense in $Q$.*

**Proof** Let $(x,h) \in 2^\omega \times \mathcal{H}$ and $(s,u) \in u$. Take $M < \omega$ such that

(1) $|s| \leq M$,

(2) if $a^{h'}_k \setminus M \neq \phi$ then $|a^{h'}_k \setminus M| > |u|$, for all $k < \omega$ and $(x',h') \in u$.

(3) if $a^h_k \setminus M \neq \phi$ then $|a^h_k \setminus M| > |u|$, for all $k < \omega$.

Using Lemma 2.1, take $(t,u) \leq (s,u)$ such that $\mathrm{dom}(t) = M$. Then, it holds that $(t, u \cup \{ (x,h) \}) \in Q$ and $(t, u \cup \{ (x,h) \}) \leq (s,u)$. $\square$

**Lemma 2.4** *$Q$ satisfies the countable chain condition.*

**Proof** Let $W$ be an uncountable subset of $Q$. Using Lemma 2.1, replace $W$ by certain stronger conditions if necessary, we may assume that, for all $(s,u) \in W$,

$$\text{for all } (x,h) \in u \text{ and } k < \omega, \text{ if } a_k^h \setminus k \neq \phi \text{ then } |a_k^h \setminus k| \geq 2|u|.$$

Take $s_0 \in 2^{<\omega}$ and $m < \omega$ such that $W' = \{(s,u) \in W \mid s = s_0 \text{ and } |u| = m\}$ is uncountable. Then, every elements in $W'$ are compatible. □

Let $\dot{G}$ be the canonical generic $Q$-name. Define $\dot{g}$ by

$$\Vdash_Q \dot{g} = \bigcup\{s \mid (s,u) \in \dot{G}, \text{ for some } u\}.$$

**Lemma 2.5** $\Vdash_Q \dot{g} \in 2^\omega$ *and* $\forall x \in 2^\omega \cap \mathbf{V}\, \forall h \in \mathcal{H} \cap \mathbf{V}\, \forall^\infty n < \omega\ \dot{g} \restriction a_n^h \neq x \restriction a_n^h$.

**Proof** This is directly followed from Lemmas 2.1 and 2.3. □

Let $\kappa$ be a regular uncountalbe cardinal and $P$ the $\kappa$-stage finite support iteration by the above forcing $Q$. Then, by the above argments, it holds that $\theta^* = \kappa$ in the generic model $\mathbf{V}^P$. Since $P$ is finite support, it adds cofinally many Cohen reals. So, in $\mathbf{V}^P$, the covering number $\mathbf{cov}(\mathcal{M})$ of the meager ideal on the real line lifts up to $\kappa$. Futhermore, the next lemma shows that the unbounding number $\mathbf{b}$ of $\omega^\omega$ lifts up to $\kappa$, too.

**Lemma 2.6** *There is a $Q$-name $\dot{d}$ such that*

$$\Vdash_Q \dot{d} \in \omega^\omega \textit{ dominates } \omega^\omega \cap \mathbf{V}.$$

**Proof** For each set $X$, denote by $\mathbf{0}_X$ the constantly zero function from $X$ to 2.

**Claim 2.7** *For any $n < \omega$,*

$$\{(s,u) \in Q \mid \exists m < \omega\ (\mathbf{0}_{[m,m+n)} \subset s)\} \textit{ is dense in } Q.$$

$\because$) Let $n < \omega$ and $(s,u) \in Q$. Take $(t,u) \leq (s,u)$ such that, for all $(x,h) \in u$ and $k < \omega$,

$$\text{if } a_k^h \setminus \mathrm{dom}(t) \neq \phi \text{ then } |a_k^h \setminus \mathrm{dom}(t)| \geq |u| + n.$$

Define $t' : |t| + n \to 2$ by $t \subset t'$ and $t'(|t| + j) = 0$, for $j < n$. It is easy to check that $(t',u) \in Q$ and $(t',u) \leq (s,u)$. △

By Claim 2.7, it holds that

$$\Vdash_Q \forall n < \omega\, \exists m < \omega\ \dot{g} \restriction [m, m+n) = \mathbf{0}_{[m,m+n)}.$$

So, in $\mathbf{V}^Q$, define $\dot{d} \in \omega^\omega$ by

$$\dot{d}(n) = \text{the smallest } m < \omega \text{ such that } n \leq m \text{ and } \dot{g} \restriction [m, m+2n) = \mathbf{0}_{[m,m+2n)}.$$

To show $\dot{d}$ is a required one, let $f \in \omega^\omega$ and $(s,u) \in Q$. Without loss of generality, we may assume that $f$ is strictly increasing. Take $h \in \mathcal{H}$ such that

$|a^h_k| \le |a^h_{k+1}|$, for all $k < \omega$ and $|\{ k < \omega \mid |a^h_k| = n \}| \ge f(n) + 1$, for all $n < \omega$. By Lemma 2.3, take $(t, v) \le (s, u)$ such that $(\mathbf{0}_\omega, h) \in v$. Let $k_0$ be the smallest $k < \omega$ such that $|t| \ge h(k)$ and set $n_0 = |a^h_{k_0}| + |t|$. The next claim completes the proof of the lemma.

**Claim 2.8** $(t, v) \Vdash_Q \forall n > n_0 \; f(n) < \dot{d}(n)$.

$\because$) To get a contradiction, assume that there are $(t', v') \le (t, v)$ and $n > n_0$ such that $(t', v') \Vdash_Q \dot{d}(n) \le f(n)$. Replace $(t', v')$ by a stronger condition if necessary, we may assume that $(t', v')$ decides the value of $\dot{d}(n)$. Let $m < \omega$ be such that $(t', v') \Vdash_Q \dot{d}(n) = m$. Without loss of generality, we may assume that $m + 2n \subset \mathrm{dom}(t')$. Let $k$ be the unique $k < \omega$ such that $m \in a^h_k$. By the choise of $h$, it holds that $|a^h_k|$, $|a^h_{k+1}| \le n$. $\therefore$ $a^h_{k+1} \subset [m, m + 2n)$. Since $(t', v') \Vdash_Q \dot{g} \restriction [m, m + 2n) = \mathbf{0}_{[m,m+2n)}$, it holds that $t' \restriction a^h_{k+1} = \mathbf{0}_{a^h_{k+1}}$. This contradicts the facts that $(t', v') \le (t, v)$ and $(\mathbf{0}_\omega, h) \in v$ and $\mathrm{dom}(t) \cap [m, m + 2n) = \phi$. □

In section 4, we give a genric model in which holds $\theta^* = \omega_2$ and $\mathbf{cov}(\mathcal{M}) = \omega_1$. But I do not known whether there is a model which satisfies $\mathbf{b} < \theta^*$.

**Question 2.1** *Is* $\mathbf{b} < \theta^*$ *consistent with* ZFC*?*

# 3 A forcing notion which lifts up $\theta$

In this section, we give a forcing notion which gives a generic model of $\theta^* = \omega_1$ and $\theta = \omega_2$. The forcing notion which we give here is constructed by the $\omega_2$-stage countable support iteration. We begin with the definition of a forcing notion $\mathbf{BT}_f$ for $f \in \mathcal{F}$ which will be used each stage in the iteration.

Let $f \in \mathcal{F}$. For each $n < \omega$, denote $\prod_{m<n} f(m)$ by $S^f_n$. Put $S^f = \bigcup_{n<\omega} S^f_n$. Note that $(S^f, \subset)$ is a tree. Define the forcing notion $(\mathbf{BT}_f, \le)$ by

$q \in \mathbf{BT}_f$

if and only if

(1) $q$ is a subtree of $S^f$.

(2) there is a function $f' \in \mathcal{F}$ such that $|\mathrm{succ}_q(s)| \ge f'(|s|)$ for every $s \in q$.

$q' \le q$ if and only if $q' \subset q$.

For each $q \in \mathbf{BT}_f$, define $\pi_q \in \omega^\omega$ by

$$\pi_q(n) = \max\{ k < \omega \mid \forall n' \ge n \, \forall s \in q \cap S^f_{n'} \; |\mathrm{succ}_q(s)| \ge k \}.$$

Note that $\pi_q \in \mathcal{F}$ for all $q \in \mathbf{BT}_f$. For each $k < \omega$, define the ordering $\leq_k$ on $\mathbf{BT}_f$ by

$q' \leq_k q$ if and only if $q' \leq q$ and $\pi_q \upharpoonright m_k = \pi_{q'} \upharpoonright m_k$,

where $m_k$ denotes the smallest $m < \omega$ such that $\pi_q(m) > k$.

In [2], Bartoszyński, Judah and Shelah have used similar but more complicated forcing notions $\mathbf{Q}_{f,g}$. The proof of the next lemma is similar to, but quite easier than the proof of Claim 2.6 in [2].

**Lemma 3.1** *Let $\dot{e}$ be a $\mathbf{BT}_f$-name such that $\Vdash \dot{e} \in \mathbf{V}$. Then, for each $k < \omega$ and $q \in \mathbf{BT}_f$, there are $q' \leq_k q$ and a finite set $E$ such that $q' \Vdash \dot{e} \in E$.*

**Proof** Let $\dot{e}$, $k < \omega$, $q \in \mathbf{BT}_f$ be as in the lemma. For each $s \in q$, denote by $q[s]$ the condition $\{\, t \in q \mid s \subset t \text{ or } t \subset s \,\}$. Take $M < \omega$ such that $\pi_q(M) \geq 2k$. Set

$$T = \{\, s \in q \mid |s| \geq M \text{ and } \exists\, q' \leq_k q[s]\, \exists\, E \;(\; E \text{ is finite and } q' \Vdash \dot{e} \in E \;) \,\}.$$

Note that, whenever $s \in q \setminus T$ and $|s| \geq M$, $|\mathrm{succ}_q(s) \cap T| < k$.

**Claim 3.2** $q \cap S^f_M \subset T$.

$\because$) To get a contradiction, assume that $s \in q \cap S^f_M \setminus T$. Let $U = \{\, t \in q \setminus T \mid s \subset t \,\}$. Then, it holds that

$$\forall t \in U \;(\; |\{\, u \in U \mid t \subset u \text{ and } |u| = |t| + 1 \,\}| > \pi_q(|u|) - k \;).$$

This implies that $r = \{\, s \upharpoonright j \mid j < |s| \,\} \cup U \in \mathbf{BT}_f$ and $r \leq_k q[s]$. Take $r' \leq r$ such that $r'$ decides $\dot{e}$. Take $t \in r'$ such that $\pi_{r'}(|t|) \geq k$. Since $r'[t] \leq_k q[t]$, we have that $t \in T$. This contradicts that $U \cap T = \phi$. $\triangle$

By Claim 3.2, for each $s \in q \cap S^f_M$, take $q_s \leq_k q[s]$ and a finite set $E_s$ such that $q_s \Vdash \dot{e} \in E_s$. Then $q' = \bigcup_{s \in q \cap S^f_M} q_s$ and $E = \bigcup_{s \in q \cap S^f_M} E_s$ satisfy this lemma. $\square$

**Corollary 3.3** $(\mathbf{BT}_f, (\leq_k)_{k<\omega})$ *satisfies Axiom A and* $\mathbf{BT}_f$ *is $\omega^\omega$-bounding.* $\square$

Let $\dot{G}$ be the canonical generic $\mathbf{BT}_f$-name. Define $\mathbf{BT}_f$-name $\dot{g}$ by

$$\Vdash \dot{g} = \bigcup (\bigcap \dot{G}) \in \prod_{n<\omega} f(n).$$

Then, it is easy to check that

$$\Vdash \forall x \in \prod_{n<\omega} f(n) \cap \mathbf{V}\; \forall^\infty n < \omega \;\; \dot{g}(n) \neq x(n).$$

Now we can describe how to construct a model which satisfies $\theta = \omega_2$ and $\theta^* = \omega_1$. Start with a ground model with CH. Let $\{\, f_\alpha \mid \alpha < \omega_2 \,\} \subset \mathcal{F}$ be such that

$\{\, \alpha < \omega_2 \mid f_\alpha = f \,\}$ is cofinal in $\omega_2$ for each $f \in \mathcal{F}$.

Define the $\omega_2$-stage countable support iteration $P_\alpha$ (for $\alpha \le \omega_2$), $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by

$$\Vdash_\alpha \ \dot{Q}_\alpha = \mathbf{BT}_{f_\alpha}.$$

Let $P = P_{\omega_2}$. Then, by the above argments, it holds that, in $\mathbf{V}^P$, $\theta = \omega_2$ and $\mathbf{d} = \omega_1$. Since $\mathrm{cof}([\omega_1]^\omega, \subset) = \omega_1$ does always hold, it holds that, in $\mathbf{V}^P$, $\theta^* \le \mathbf{d} = \omega_1$.

# 4 A generic model of $\theta = \omega_2$ and $\mathbf{cov}(\mathcal{M}) = \omega_1$

In the previous section, we show that $\mathbf{BT}_f$ does not lift up $\theta^*$. But, if we first add a dominating real then we get a certain function $f \in \mathcal{F}$ such that $\mathbf{BT}_f$ lifts up $\theta^*$. In this section, we show that $\theta^*$ can be separeted from $\mathbf{cov}(\mathcal{M})$ by using it.

**Lemma 4.1** *Let $\mathbf{V}$, $\mathbf{W}$ be transitive models of* ZFC *such that $\mathbf{V} \subset \mathbf{W}$. Assume that $d \in \mathbf{W} \cap \omega^\omega$ dominates $\mathbf{V} \cap \omega^\omega$. In $\mathbf{W}$, define $h \in \mathcal{H}$ by*

$$|a^h_k| \le |a^h_{k+1}|, \text{ for all } k < \omega \text{ and } |\{ k < \omega \mid |a^h_k| = n \}| = d(n) + 1, \text{ for all } n < \omega.$$

*Then, it holds that $\forall^\infty m < \omega \, \exists k < \omega \ a^h_k \subset a^{h'}_m$ for all $h' \in \mathbf{V} \cap \mathcal{H}$.*

**Proof** Let $h' \in \mathbf{V} \cap \mathcal{H}$. In $\mathbf{V}$, define $f_0, f_1 \in \omega^\omega$ by

$$f_0(n) = \text{the smallest } m < \omega \text{ such that } \forall m' \ge m \ \ |a^{h'}_{m'}| \ge 2n, \text{ and}$$

$$f_1(n) = \max \ a^{h'}_{f_0(n+1)}.$$

Since $d$ dominates $f_0$, $f_1$, there is $n_0 < \omega$ such that $\forall n \ge n_0 \ f_0(n), f_1(n) < d(n)$. Put $k_0 = f_0(n_0)$. To show that $\forall k \ge k_0 \, \exists j < \omega \ a^h_j \subset a^{h'}_k$, let $k \ge k_0$. Take $n < \omega$ such that $f_0(n) \le k < f_0(n+1)$. Then, it holds that $|a^{h'}_k| \ge 2n$ and $\max a^{h'}_k < \max a^{h'}_{f_0(n+1)} = f_1(n) \le d(n)$. Since $[0, d(n))$ is covered by $\{ a^h_j \mid j < d(n) \}$ and $|a^h_j| \le n$ for all $j < d(n)$, there is $j < d(n)$ such that $a^h_j \subset a^{h'}_k$. □

**Lemma 4.2** *Let $\mathbf{V}$, $\mathbf{W}$, $d$ and $h$ be as in Lemma 4.1. Working in $\mathbf{W}$. Define $f \in \mathcal{F}$ by*

$$f(k) = 2^{|a^h_k|}, \text{ for all } k < \omega.$$

*Then, there is a $\mathbf{BT}_f$-name $\dot{y}$ such that*

$$\Vdash \dot{y} \in 2^\omega \text{ and } \Vdash \forall^\infty k < \omega \ \dot{y} \restriction a^{h'}_k \ne x \restriction a^{h'}_k, \text{ for all } x \in 2^\omega \cap \mathbf{V} \text{ and } h' \in \mathcal{H} \cap \mathbf{V}.$$

**Proof** Working in $\mathbf{W}$. Considering bijections from $f(k)$ to ${}^{a^h_k}2$ for $k < \omega$, we may identfy $\prod_{k<\omega} f(k)$ with $\prod_{k<\omega} {}^{a^h_k}2$. Let $\dot{G}$ be the canonical generic $\mathbf{BT}_f$-name. Define $\mathbf{BT}_f$-names $\dot{g}$ and $\dot{y}$ by

$$\Vdash \dot{g} = \bigcup(\bigcap \dot{G}) \text{ and } \dot{y} = \bigcup_{k<\omega} \dot{g}(k).$$

Note that $\Vdash \dot{g} \in \prod_{k<\omega} {}^{a_k^h}2$ and $\dot{y} \in 2^\omega$. It is easy to check that

$$\Vdash \forall x \in 2^\omega \cap \mathbf{W} \, \forall^\infty k < \omega \;\; \dot{y} \restriction a_k^h \neq x \restriction a_k^h.$$

To show $\dot{y}$ is as required, let $x \in \mathbf{V} \cap 2^\omega$ and $h' \in \mathbf{V} \cap \mathcal{H}$. Since it holdds that $x \in \mathbf{W}$ and $\forall^\infty m < \omega \, \exists \, k < \omega \; a_k^h \subset a_m^{h'}$, we have that

$$\Vdash \forall^\infty m < \omega \;\; \dot{y} \restriction a_m^{h'} \neq x \restriction a_m^{h'}. \qquad \square$$

**Corollary 4.3** *Assume that* CH *holds. There are a forcing notion $R$ and $R$-name $\dot{y}$ such that*

(1) *$R$ is proper and does not add a Cohen real and $|R| = \omega_1$.*

(2) $\Vdash_R \; \dot{y} \in 2^\omega$ *and* $\forall x \in 2^\omega \cap \mathbf{V} \, \forall h \in \mathcal{H} \cap \mathbf{V} \, \forall^\infty k < \omega \, \dot{y} \restriction a_k^h \neq x \restriction a_k^h.$ $\square$

Using Corollary 4.3, we can construct a generic model which satisfies $\mathbf{cov}(\mathcal{M}) = \omega_1$ and $\theta^* = \omega_2$. Start with a ground model with CH. Take an $\omega_2$-stage countable support iteration by the forcing notion as in Corollary 4.3. Since the iteration does not add a Cohen real, $\mathbf{cov}(\mathcal{M})$ remains $\omega_1$. On the other hand, since functions $\dot{y} \in 2^\omega$ which satisfy (2) in the corollary is added cofinally, $\theta^*$ must be lifted up.

# References


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